# On Mathematical Analysis of Synchronization for Bidirectionally Coupled Kuramoto Oscillators

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# Outline

# Kuramoto model with uniform coupling strength

The Kuramoto model reads

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i), \ t > 0, \ i = 1, 2, \cdots, N. \tag{1.1}$$

$\theta_i : \mathbb{R}^+ \to \mathbb{R}$ phase function, $\dot{\theta}_i = \frac{d\theta_i}{dt}$ frequency (angular velocity), $\omega_i$: natural frequency, $K > 0$: coupling strength, $N$: number of oscillators.

- Juan A. Acebrón: survey paper (Reviews of Modern Physics 2005).
- Seung-Yeal Ha et al: sufficient conditions for synchronization (Physica D, 2010), orbital stability (Physica D, 2012), improved results (Nonlinearity, 2015), generic initial configuration in a large coupling regime (Commun. Math. Sci., 2016), etc.

# Kuramoto model with uniform coupling strength

- Chun-Hsiung Hsia et al:
  - 2nd order Kuramoto model (Journal of Differential Equations, 2019)

$$m\ddot{\theta}_i + \dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) \quad t > 0.$$

  - effects of time delay and phase lag (Journal of Differential Equations, 2020)

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j(t-\tau_{ij}) - \theta_i(t) + \gamma_{ij}), \quad t > \max\{\tau_{ij}\},$$

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j(t-\tau_{ij}) - \theta_i(t-\tau_{ij}) + \gamma_{ij}), \quad t > \max\{\tau_{ij}\}.$$

  - global convergence for frequency synchronization for Kuramoto model when $N = 3$ (Discrete and Continuous Dynamical System, 2020).

# Definition of Phase Synchronization & Frequency Synchronization

## Definition (Phase Synchronization)

We say $\{\theta_i(t)\}_{i=1}^N$ achieves a phase synchronization asymptotically if for every $i, j \in \{1, 2, \cdots, N\}$, there exist integer $n_{ij}$ such that

$$\lim_{t\to\infty} (\theta_i(t) - \theta_j(t) - 2n_{ij}\pi) = 0.$$

## Definition (Frequency Synchronization)

We say $\{\theta_i(t)\}_{i=1}^N$ achieves a frequency synchronization asymptotically if

$$\lim_{t\to\infty} (\dot{\theta}_i(t) - \dot{\theta}_j(t)) = 0$$

for all $i, j \in \{1, 2, \cdots, N\}$.

# Mean Zero Condition on Natural Frequency

Denote the vector-valued function

$$\Theta(t) := (\theta_1(t), \cdots, \theta_N(t)) \in \mathbb{R}^N, \qquad \Omega := (\omega_1, \cdots, \omega_N) \in \mathbb{R}^N,$$

and the diameter function

$$D(\Theta(t)) := \max_{i \neq j}\{\theta_i(t) - \theta_j(t)\}, \qquad D(\Omega) := \max_{i \neq j}\{\omega_i - \omega_j\}.$$

Denote the mean of natural frequencies by $\overline{\omega} = \frac{1}{N}\sum_{i=1}^N \omega_i$. Note that Kuramoto model (1.1) is invariant under the change of variables: $\hat{\theta}_i(t) = \theta_i(t) - \overline{\omega}t$ and $\hat{\omega}_i = \omega_i - \overline{\omega}$. Hence, without loss of generality, we assume that

$$\sum_{i=1}^N \omega_i = 0. \tag{1.2}$$

# Mean Zero Condition on Natural Frequency

By taking summation of (1.1) over $i = 1, 2, \cdots, N$, we have

$$\sum_{i=1}^{N} \dot{\theta}_i = \sum_{i=1}^{N} \omega_i = 0 \Rightarrow \sum_{i=1}^{N} \theta_i(t) \text{ is invariant for } t > 0.$$

It implies the following lemma:

**Lemma**

*Under the condition* (1.2),

$$\sup_{t>0} |\theta_i(t)| < \infty \textit{ for all } i \Rightarrow \sup_{t>0} D(\Theta(t)) < \infty.$$

By multiplying (1.1) with $\dot{\theta}_i$, summing all equations and integrating the resulting equation with respect to time variable over $[0,t]$, we obtain that

$$\begin{aligned} &\int_0^t \sum_{i=1}^N \dot{\theta}_i(s)^2\,ds \\ &= \int_0^t \sum_{i=1}^N \omega_i \dot{\theta}_i(s)\,ds + \int_0^t \frac{K}{2}\sum_{i=1}^N \cos(\theta_j(s)-\theta_i(s))\dot{\theta}_i(s)\,ds \\ &= \sum_{i=1}^N \omega_i\theta_i(t) - \underbrace{\sum_{i=1}^N \omega_i\theta_i(0)}_{\text{fixed value}} + \underbrace{\frac{K}{2}\sum_{i<j}^N \Big(\cos(\theta_i(t)-\theta_j(t)) - \cos(\theta_i(0)-\theta_j(0))\Big)}_{\text{bounded for } t>0}. \end{aligned} \tag{1.3}$$

# Lyapunov Function

By combining (1.3) and the uniform continuity of $\dot{\theta}_i$, we have the following lemma:

## Lemma

*Let* $\Theta(t)$ *be the solution of* (1.1)*. Suppose* $\sup_{t>0} |\theta_i(t)| < \infty$ *for all* $i = 1, 2, \cdots, N$*. Then we have*

$$\lim_{t\to\infty} \dot{\theta}_i(t) = 0 \quad \text{for all } i = 1, 2, \cdots, N. \tag{1.4}$$

*That is,* $\Theta(t)$ *achieves a frequency synchronization asymptotically.*

## Remark

Under the mean zero condition (1.2), the frequency synchronization is equivalent to (1.4).

# Derivation of Synchronization for Kuramoto Model

Suppose that $D(\Theta(t)) = \theta_i(t) - \theta_j(t) < \pi$ for some $i, j$ and $t > 0$. Then

$$\begin{aligned}\dot{\theta}_i(t) - \dot{\theta}_j(t) &= \omega_i - \omega_j + \frac{K}{N}\sum_{l=1}^{N}\Big(\sin(\theta_l(t) - \theta_i(t)) - \sin(\theta_l(t) - \theta_j(t))\Big)\\ &= \omega_i - \omega_j - \frac{2K}{N}\underbrace{\sin\Big(\frac{\theta_i(t) - \theta_j(t)}{2}\Big)}_{>0}\underbrace{\sum_{l=1}^{N}\cos\Big(\theta_l(t) - \frac{\theta_i(t) + \theta_j(t)}{2}\Big)}_{>0}.\end{aligned}$$

We can pick up $K$ is large enough such that $\dot{\theta}_i(t) - \dot{\theta}_j(t) < 0$. It implies that $D(\theta(t))$ is bounded for $t > 0$ which guarantees the frequency synchronization.

# Well-known Results of Synchronization for Kuramoto Model

Therefore, we have the following result of frequency synchronization.

**Theorem (Frequency Synchronization for (1.1))**

*Given* $\alpha \in (0, \frac{\pi}{2})$. *Suppose* $D(\Omega) < K \sin\alpha$. *Let* $\Theta(t)$ *be the solution of* $(1.1)$. *If the initial configuration satisfies* $D(\Theta(0)) < \pi - \alpha$, *then we have*

1. $D(\Theta(t)) < \pi - \alpha \quad$ *for all* $t > 0$;
2. $\Theta(t)$ *achieves a frequency synchronization asymptotically.*

For the identical case, we have the following result of phase synchronization.

**Theorem (Phase Synchronization for (1.1))**

*Suppose* $\omega_i = \omega$ *for all* $i = 1, 2, \cdots, N$. *Let* $\Theta(t)$ *be the solution of* $(1.1)$. *If the initial configuration satisfies* $D(\Theta(0)) < \pi$, *then* $\Theta(t)$ *achieve a phase synchronization asymptotically.*

# Kuramoto model with non-uniform coupling strength

The Kuramoto model with non-uniform coupling strength is considered as

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} k_{ij} \sin(\theta_j - \theta_i), \; t > 0, \; i = 1, 2, \cdots, N. \tag{1.5}$$

- Florian Dörfler et al: multirate Kuramoto model (SIAM J. Applied Dynamical System 2011), non-uniform coefficients (SIAM J. Control and Optimization, 2012), application of networks and smart grids (PNAS, 2013), survey paper (Automatica, 2014), etc.
- J.-G. Dong and X. Xue: general connectivity and dampings (Commun. Math. Sci., 2013).
- S-Y. Ha et al: $L^2$-energy method (Journal of Differential Equations, 2013).
- S-H. Chen et al: involve the control of pacemaker (IEEE Transactions on Circuits and Systems–I: Regular papers, 2021)

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} k_{ij} \sin(\theta_j - \theta_i) + F_i \sin(\sigma t - \theta_i), \; t > 0, \; i = 1, 2, \cdots, N.$$

# Bidirectionally Coupled Kuramoto Model

In this talk, we consider the following bidirectionally coupled Kuramoto model:

$$\dot{\theta}_i = \omega_i + \frac{K}{2}\Big(\sin(\theta_{i+1} - \theta_i) + \sin(\theta_{i-1} - \theta_i)\Big), \ t > 0, \ i = 1, 2, \dots, N. \tag{1.6}$$

We adopt that $\theta_0 = \theta_N$ and $\theta_{N+1} = \theta_1$. Without loss of generality, we have the mean zero condition on natural frequency:

$$\sum_{i=1}^{N} \omega_i = 0. \tag{1.7}$$

Application: concatenation in power system, Josephson junction array, networks, neuroscience, etc.

# Synchronization Theory for Identical Case

**Theorem (Global Convergence for Frequency Synchronization)**

*Suppose* $\omega_i = \omega_j$ *for all* $i \neq j$. *Let* $\Theta(t)$ *be the solution of* (1.6). *Then we have*

$$\lim_{t \to \infty} \dot{\theta}_i(t) = 0 \quad \text{for all } i = 1, 2, \cdots, N.$$

Note that this result is independent of the initial configuration. It is referred to as unconditional frequency synchronization.

*Proof*: Apply the mean zero assumption (1.7) and Lyapunov function. □

# Synchronization Theory for Identical Case

## Theorem (Phase Synchronization)

*Suppose* $\omega_i = \omega_j$ *for all* $i \neq j$*. Let* $\Theta(t)$ *be the solution of* $(1.6)$ *with the initial condition satisfying*

$$0 < \theta_i(0) < \pi \qquad \text{for all } i = 1, 2, \cdots, N.$$

*Then there exists a constant* $c \in (0, \pi)$ *such that*

$$\lim_{t \to \infty} \theta_i(t) = c \qquad \text{for all } i = 1, 2, \cdots, N.$$

*Proof*:

1. Show that $\theta_M$ is decreasing and $\theta_m$ is increasing, where

$$\theta_M(t) = \max_{1 \le i \le N} \theta_i(t), \text{ and } \theta_m(t) = \min_{1 \le i \le N} \theta_i(t).$$

2. Hence, $\lim_{t \to \infty} \theta_M(t) = c_1$, $\lim_{t \to \infty} \theta_m(t) = c_2$, and $0 < c_2 \le c_1 < \pi$. Show that $c_1 = c_2$. □

# Synchronization Theory for Identical Case

## Theorem (Phase Synchronization)

*Suppose* $\omega_i = \omega_j$ *for all* $i \neq j$. *Let* $\Theta(t)$ *be the solution of* (1.6) *with the initial condition satisfying*

$$|\theta_{i+1}(0) - \theta_i(0)| < \frac{\pi}{2} \quad \text{for all } i = 1, 2, \cdots, N.$$

*Then we have*

$$\lim_{t\to\infty} |\theta_{i+1}(t) - \theta_i(t)| = 0 \quad \text{for all } i = 1, 2, \cdots, N.$$

*Proof*: There is $\alpha \in (0, \frac{\pi}{2})$ such that $|\theta_{i+1}(0) - \theta_i(0)| < \frac{\pi}{2} - \alpha$ for all $i$. Claim that

$$|\theta_{i+1}(t) - \theta_i(t)| < \frac{\pi}{2} - \alpha \quad \text{for all } t > 0,\ i = 1, 2, \cdots, N.$$

Then we can conclude the desired result. □

# Frequency Synchronization for Non-identical Case

Due to the previous observations, in order to obtain the frequency synchronization, we need to develop certain conditions such that $D(\Theta(t))$ is bounded. Precisely, we shall show that

$$D(\Theta(t)) < \frac{\pi}{2} - \alpha \quad \text{for all } t > 0$$

with the initial condition satisfying $D(\Theta(0)) < \gamma$, where $\gamma < \frac{\pi}{2} - \alpha$ needs to be determined.

**Lemma**

*For any $A, B, C \in \mathbb{R}$, we have*

$$\sin A + \sin B + \sin C = \sin(A + B + C) + 4 \sin(\frac{A + B}{2}) \sin(\frac{B + C}{2}) \sin(\frac{A + C}{2}). \quad (2.1)$$

Suppose $t_0 > 0$ is the first moment that $D(\Theta(\cdot))$ hits $\frac{\pi}{2} - \alpha$. Assume that $D(\Theta(t_0)) = \theta_{i+k}(t_0) - \theta_i(t_0)$ for some $i \in \{1, 2, \cdots, N\}$ and $k \in \{1, 2, \cdots, [\frac{N}{2}]\}$.

# Frequency Synchronization for Non-identical Case

It is clear that $\dot{\theta}_{i+k}(t_0) - \dot{\theta}_i(t_0) \geq 0$. On the other hand, by (1.6),

$$
\begin{aligned}
&\dot{\theta}_{i+k}(t_0) - \dot{\theta}_i(t_0) \\
&\leq D(\Omega) + \frac{K}{2}\Big(\underbrace{\sin(\theta_{i+k-1}(t_0) - \theta_{i+k}(t_0))}_{\leq 0} + \sin(\theta_{i+k-1}(t_0) - \theta_{i+k}(t_0)) \\
&\qquad - \sin(\theta_{i+1}(t_0) - \theta_i(t_0)) - \underbrace{\sin(\theta_{i-1}(t_0) - \theta_i(t_0))}_{\geq 0}\Big).
\end{aligned}
$$

Inferring from (2.1), we have

$$
\begin{aligned}
&\sin(\theta_{i+k-1} - \theta_{i+k}) - \sin(\theta_{i+1} - \theta_i) \\
&= -\sin(\theta_{i+k} - \theta_{i+k-1}) + \sin(\theta_{i+k-1} - \theta_{i+1}) - \sin(\theta_{i+k-1} - \theta_{i+1}) - \sin(\theta_{i+1} - \theta_i) \\
&= \sin(\theta_{i+k-1} - \theta_{i+1}) - \sin(\theta_{i+k} - \theta_i) \\
&\quad - 4\sin(\frac{\theta_{i+k} - \theta_{i+1}}{2})\sin(\frac{\theta_{i+k-1} - \theta_i}{2})\sin(\frac{\theta_{i+k} - \theta_i}{2} - \frac{\theta_{i+k-1} - \theta_{i+1}}{2}).
\end{aligned}
$$

# Frequency Synchronization for Non-identical Case

Note that

$$0 \le \frac{\theta_{i+k}(t_0) - \theta_{i+1}(t_0)}{2} \le \frac{\pi}{4} - \frac{\alpha}{2},$$
$$0 \le \frac{\theta_{i+k-1}(t_0) - \theta_{i}(t_0)}{2} \le \frac{\pi}{4} - \frac{\alpha}{2},$$
$$0 \le \frac{\theta_{i+k}(t_0) - \theta_{i}(t_0)}{2} - \frac{\theta_{i+k-1}(t_0) - \theta_{i+1}(t_0)}{2} \le \frac{\pi}{2} - \alpha.$$

Hence,

$$\dot{\theta}_{i+k}(t_0) - \dot{\theta}_i(t_0) \le D(\Omega) + \frac{K}{2}\left( \sin(\theta_{i+k-1}(t_0) - \theta_{i+1}(t_0)) - \cos\alpha \right).$$

Therefore, it guarantees the following essential lemma:

# Frequency Synchronization for Non-identical Case

## Lemma

*Assume* $D(\Theta(t_0)) = \frac{\pi}{2} - \alpha$ *and* $D(\Theta(t)) < \frac{\pi}{2} - \alpha$ *for all* $t \in [0, t_0)$. *Suppose* $D(\Theta(t_0)) = |\theta_{i+k}(t_0) - \theta_i(t_0)| = \frac{\pi}{2} - \alpha$ *for some* $i \in \{1, 2, \cdots, N\}$ *and* $k \in \{1, 2, \cdots, [N/2]\}$.

1. *If* $\theta_{i+k}(t_0) - \theta_i(t_0) > 0$, *we have*

$$\dot{\theta}_{i+k}(t_0) - \dot{\theta}_i(t_0) \leq D(\Omega) + \frac{K}{2}\Big(\sin(\theta_{i+k-1}(t_0) - \theta_{i+1}(t_0)) - \cos\alpha\Big). \tag{2.2}$$

2. *If* $\theta_{i+k}(t_0) - \theta_i(t_0) < 0$, *we have*

$$\dot{\theta}_i(t_0) - \dot{\theta}_{i+k}(t_0) \leq D(\Omega) + \frac{K}{2}\Big(\sin(\theta_{i+1}(t_0) - \theta_{i+k-1}(t_0)) - \cos\alpha\Big). \tag{2.3}$$

We need to construct the boundedness of $|\theta_{i+k-1}(t_0) - \theta_{i+1}(t_0)|$. To do this, we have to study the dynamics of $|\theta_{i+k-1} - \theta_{i+1}|$ and inductively get a recursive formula.

# Frequency Synchronization for Non-identical Case

In case that $D(\Theta(t_0)) = \theta_{i+k}(t_0) - \theta_i(t_0)$ for $k = 1, 2$,

$$\dot{\theta}_{i+k}(t_0) - \dot{\theta}_i(t_0) \leq D(\Omega) + \frac{K}{2}\Big(\sin(\theta_{i+k-1}(t_0) - \theta_{i+k}(t_0) - \sin(\theta_{i+1}(t_0) - \theta_i(t_0))\Big)$$

$$= \begin{cases} D(\Omega) - K\cos\alpha, & \text{if } k = 1 \\ D(\Omega) - \dfrac{K}{2}\cos\alpha, & \text{if } k = 2 \end{cases}$$

$$\leq D(\Omega) - \frac{K}{2}\cos\alpha.$$

In case that $D(\Theta(t_0)) = \theta_{i+3}(t_0) - \theta_i(t_0)$,

$$\dot{\theta}_{i+3}(t_0) - \dot{\theta}_i(t_0) \leq D(\Omega) + \frac{K}{2}\Big(\sin(\theta_{i+2}(t_0) - \theta_{i+1}(t_0)) - \cos\alpha\Big).$$

# Frequency Synchronization for Non-identical Case

Note that, for $t \in (0, t_0)$,

$$\begin{aligned}
\dot{\theta}_{i+2} - \dot{\theta}_{i+1} &= \omega_{i+2} - \omega_{i+1} - K\sin(\theta_{i+2} - \theta_{i+1}) \\
&\quad + K\cos\left(\frac{\theta_{i+3} + \theta_i}{2} - \frac{\theta_{i+2} + \theta_{i+1}}{2}\right)\sin\left(\frac{\theta_{i+3} - \theta_i}{2} - \frac{\theta_{i+2} - \theta_{i+1}}{2}\right) \\
&\le D(\Omega) + K\sin\left(\frac{\frac{\pi}{2} - \alpha}{2} - \frac{\theta_{i+2} - \theta_{i+1}}{2}\right) - K\sin(\theta_{i+2} - \theta_{i+1}).
\end{aligned}$$

Therefore, if $D(\Omega)$ and $D(\Theta(0))$ satisfy some certain conditions, then we can derive that

$$\theta_{i+2}(t_0) - \theta_{i+1}(t_0) \le \beta_1$$

for some $\beta_1 \in (0, \frac{\pi}{2} - \alpha)$. Hence, we get that

$$\dot{\theta}_{i+3}(t_0) - \dot{\theta}_i(t_0) \le D(\Omega) + \frac{K}{2}(\sin\beta_1 - \cos\alpha).$$

# Frequency Synchronization for Non-identical Case

In case that $D(\Theta(t_0)) = \theta_{i+4}(t_0) - \theta_i(t_0)$.

$$\dot{\theta}_{i+4}(t_0) - \dot{\theta}_i(t_0) \le D(\Omega) + \frac{K}{2}\Big(\sin(\theta_{i+3}(t_0) - \theta_{i+1}(t_0)) - \cos\alpha\Big).$$

Note that, for $t \in (0, t_0)$,

$$\begin{aligned}
\dot{\theta}_{i+3} - \dot{\theta}_{i+1} &= \omega_{i+3} - \omega_{i+1} \\
&\quad + K\cos\Big(\frac{\theta_{i+4} + \theta_i}{2} - \frac{\theta_{i+3} + \theta_{i+1}}{2}\Big)\sin\Big(\frac{\theta_{i+4} - \theta_i}{2} - \frac{\theta_{i+3} - \theta_{i+1}}{2}\Big) \\
&\quad - K\cos\Big(\theta_{i+2} - \frac{\theta_{i+3} + \theta_{i+1}}{2}\Big)\sin\Big(\frac{\theta_{i+3} - \theta_{i+1}}{2}\Big) \\
&\le D(\Omega) + K\sin\Big(\frac{\frac{\pi}{2} - \alpha}{2} - \frac{\theta_{i+3} - \theta_{i+1}}{2}\Big) - K\sin\alpha\sin\Big(\frac{\theta_{i+3} - \theta_{i+1}}{2}\Big).
\end{aligned}$$

# Frequency Synchronization for Non-identical Case

Therefore, if $D(\Omega)$ and $D(\Theta(0))$ satisfy some certain conditions, then we can derive that

$$\theta_{i+3}(t_0) - \theta_{i+1}(t_0) \leq \beta_2$$

for some $\beta_2 \in (0, \frac{\pi}{2} - \alpha)$. Hence, we get that

$$\dot{\theta}_{i+3}(t_0) - \dot{\theta}_i(t_0) \leq D(\Omega) + \frac{K}{2}(\sin\beta_2 - \cos\alpha).$$

In summary, there exist parameters $z_{N,\alpha}$ and $\beta_1, \cdots, \beta_{[\frac{N}{2}]-2}$ for the case $N > 5$. Then we have the main result of frequency synchronization for the non-identical case.

# Frequency Synchronization for Non-identical Case

## Theorem (Frequency Synchronization)

*Suppose* $\alpha \in (0, \frac{\pi}{2})$. *Assume that*

$$\frac{D(\Omega)}{K} < \mu(N,\alpha) := \begin{cases} \frac{1}{2}\cos\alpha, & \textit{if } N \le 5, \\ \textit{min}\left\{ z_{N,\alpha}, \frac{1}{2}(\cos\alpha - \sin\beta_{[\frac{N}{2}]-2}) \right\}, & \textit{if } N > 5. \end{cases}$$

*Suppose* $\Theta(t)$ *is the solution of* (1.6) *with the mean zero assumption on natural frequencies* (1.7). *If the initial configuration satisfies*

$$D(\Theta(0)) < \gamma(N,\alpha) := \begin{cases} \frac{\pi}{2} - \alpha, & \textit{if } N \le 5, \\ \beta_1, & \textit{if } N > 5. \end{cases}$$

*then we have*

1. $D(\Theta(t)) < \frac{\pi}{2} - \alpha$ *for all* $t > 0$, *and*
2. $\lim_{t\to\infty} \dot{\theta}_i(t) = 0$ *for all* $i = 1, 2, \cdots, N$.

# Frequency Synchronization for Non-identical Case

## Theorem (Frequency Synchronization for $N \le 5$ (Improved Result))

*Suppose* $N \le 5$ *and* $\alpha \in (0, \frac{\pi}{2})$. *Assume that*

$$D(\Omega) \le \frac{K}{2} \sin\alpha.$$

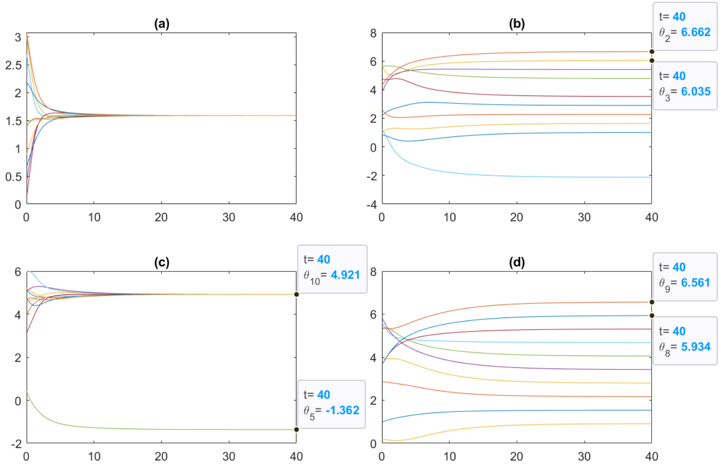
*Let* $\Theta(t)$ *be the solution of* (1.6) *with the mean zero assumption on natural frequencies* (1.7). *If the initial configuration satisfies*

$$D(\Theta(0)) < \pi - \alpha,$$

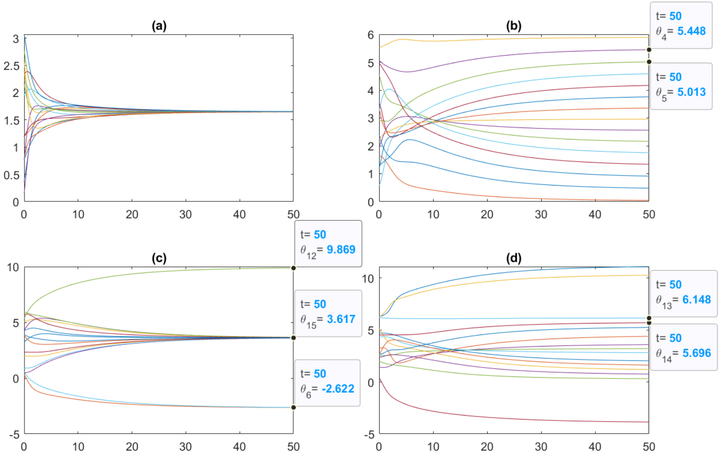
*then we have*

1. $D(\Theta(t)) < \pi - \alpha$ *for all* $t > 0$, *and*
2. $\lim_{t\to\infty} \dot{\theta}_i(t) = 0$ *for all* $i = 1, 2, \cdots, N$.

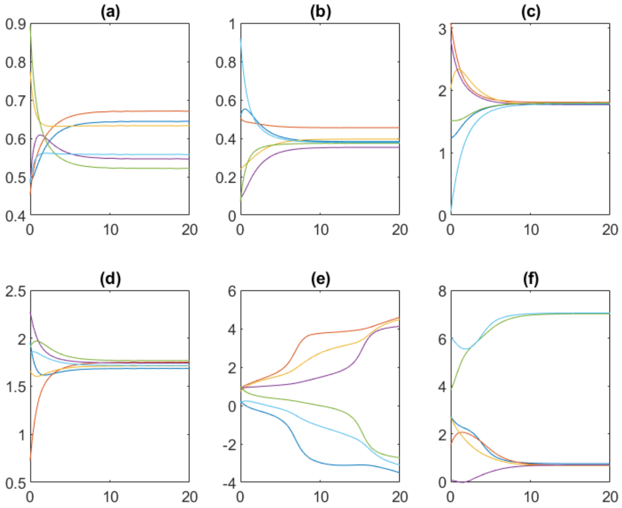
# Set $N = 10$ and $\omega = 0$ in (1.6)



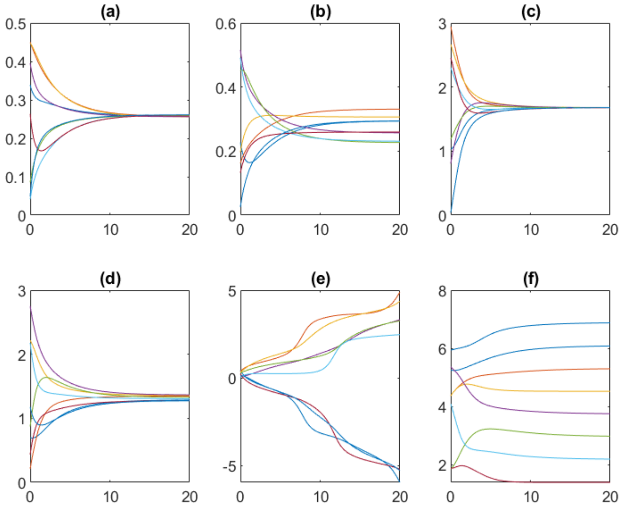
# Set $N = 15$ and $\omega = 0$ in (1.6)



# Set $N = 6$, $\alpha = 0.1$, $z_{6,0.1} = 0.6450$, $\mu = 0.0672$, $\gamma = 1.0363$



# Set $N = 8$, $\alpha = 0.1$, $z_{8,0.1} = 0.0370$, $\mu = 0.0036$, $\gamma = 0.5107$



# Summary

We consider the first order bidirectionally coupled Kuramoto model:

$$\dot{\theta}_i = \omega_i + \frac{K}{2}\Big(\sin(\theta_{i+1} - \theta_i) + \sin(\theta_{i-1} - \theta_i)\Big),\ t > 0,\ i = 1, 2, \ldots, N.$$

1. For identical case, this model exhibits the global convergence for frequency synchronization. If the initial configuration is restricted in certain regime, these oscillators achieve a phase synchronization asymptotically.
2. For non-identical case, this model exhibits the frequency synchronization if natural frequency, coupling strength, and initial configuration satisfy $\frac{D(\Omega)}{K} < \mu$ and $D(\Theta(0)) < \gamma$, respectively. Moreover, $\gamma$ has a positive lower bound.

# Future Work

1. Improve the sufficient conditions in the theorems of frequency synchronization for the non-identical case.
2. Investigate the theoretical result of frequency synchronization for the unidirectionally coupled Kuramoto model, i.e.

$$\dot{\theta}_i = \omega_i + K \sin(\theta_{i+1} - \theta_i), \ t > 0, \ i = 1, 2, \cdots, N.$$

   The result of phase synchronization for the identical case is studied by S-Y Ha in SIAM J. Appl. Math., 2012.
3. Involve the effect of time delay in the bidirectionally coupled Kuramoto model, i.e.

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{2}\Big(\sin\left(\theta_{i+1}(t-\tau) - \theta_i(t)\right) + \sin\left(\theta_{i-1}(t-\tau) - \theta_i(t)\right)\Big),$$

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{2}\Big(\sin\left(\theta_{i+1}(t-\tau) - \theta_i(t-\tau)\right) + \sin\left(\theta_{i-1}(t-\tau) - \theta_i(t-\tau)\right)\Big).$$

# Future Work

4. Consider the general coupling function $\Gamma(\theta)$ in the 1st order bidirectionally coupled Kuramoto model, i.e.

$$\dot{\theta}_i = \omega_i + \frac{K}{2}\Big(\Gamma(\theta_{i+1} - \theta_i) + \Gamma(\theta_{i-1} - \theta_i)\Big),$$

where $\Gamma$ is an odd, periodic function and has some certain properties.

Thanks for your attention!